NONLINEAR OSCILLATIONS IN A COLD PLASMA*

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ABSTRACT

In a Maxwell-fluid description large amplitude electrostatic and electromagnetic oscillations in a cold plasma are analyzed in situations where the spatial variations are one-dimensional and the ions form a fixed neutralizing background. In the electrostatic approximation a Lagrangian description following the electron motion is adopted, and exact solutions obtained in these variables in situations where multistream flow does not develop. The inversion to Eulerian coordinates is carried out explicitly for the particular example of an initial sinusoidal perturbation in density. A model describing the dispersive modifications of (weak) thermal effects is analyzed showing the phase-mixing of (moderately) large amplitude initial disturbances. After sufficient time, the electron fluid becomes stationary and a low level of electric field remains balancing the force due to pressure variations. In

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addition, for the case of nonlinear electrostatic oscillations perpendicular to a static magnetic field, the exact solution in Lagrangian variables shows that for a cold plasma coherent oscillations at the upper hybrid frequency are maintained indefinitely over the regions of initial excitation.

Electromagnetic oscillations are considered as developing from small initial values on the background of the large-amplitude longitudinal oscillations already calculated. The resulting wave equation is analyzed for several limiting cases. It is shown that for sufficiently short wavelengths of the transverse fields there exists an infinite number of narrow ranges of the wave number in which the transverse oscillations are unstable. The growth is ultimately limited by the magnetic force which was neglected in the description of the longitudinal motion.

Finally, stationary solutions are studied. In the electrostatic approximation a very special class of periodic Bernstein-Greene-Kruskal waves without trapped particles is obtained. Again the electromagnetic solutions are unstable (in space), the growth being limited by the same effect as in the time-dependent problem. A simple class of special stationary solutions is obtained for the complete problem including all magnetic force terms.

I. INTRODUCTION

It is the purpose of the present article to consider in some detail the problem of large-amplitude electrostatic and electromagnetic oscillations in a cold plasma in situations where the spatial variations are one-dimensional, and the ions form a fixed neutralizing background. The nonlinear electrostatic problem of Dawson[1] and Kalman[2] is briefly formulated in Section II-(a) introducing Lagrangian variables following the electron fluid, and the exact solutions for relevant physical quantities are obtained in those variables. Throughout the present paper, analysis is restricted to initial-value problems for which multi-stream flow does not develop. In Section II-(b), a particular example is considered in which the electron fluid is initially at rest, and the electron density has the form,

$$n(x_0,0) = n_0(1+\Delta\cos kx_0) \ , \qquad (1.1)$$

where n_0 is the uniform ion background density. The inversion to Eulerian variables is carried out explicitly and exact nonlinear expressions for the velocity, electric field, and density in the laboratory frame are obtained. The relation between these solutions and the solution to the Vlasov equation with appropriate (cold) initial conditions is briefly discussed.

Various extensions of the electrostatic problem, and the inclusion of electromagnetic effects, are considered in Sections III and IV, respectively. In Section III-(a) the analysis of II-(a) is extended to describe large-amplitude electrostatic oscillations perpendicular to a static magnetic field, and the exact solution is obtained in Lagrangian variables. In this case, coherent oscillations at the upper hybrid frequency are maintained indefinitely over the region of initial excitation and remain local to that region. A model relevant to describing the dispersive effects of small, but finite, plasma temperature is analysed in Section III-(b). Coherent oscillations at the plasma frequency are no longer maintained for all time as in the cold plasma considerations of II-(a). It is shown that for absolutely integrable initial disturbances the electron velocity phase mixes to zero; however, a low level of electric field, balancing the force due to pressure variations, remains in the time-asymptotic limit. In Section III-(c) the analysis of II-(a) is modified by the inclusion of a collisional drag term in the equation of motion for the electron fluid. For arbitrary, large-amplitude, initial disturbances (which do not lead to multistream flow), this dissipation leads to a uniform, field-free, stationary, equilibrium. For clarity, the various generalizations of the electrostatic problem considered in Section III have been treated separately; it should be noted, however, that nothing restricts the effects of static magnetic field, dispersion and dissipation from being included in a single Lagrangian analysis.

In Section IV the development of transverse fields is considered in detail. The solutions of the electrostatic problem enter as known coefficients depending on space and time in the wave equation. Several limiting cases are treated. In one of these, the solutions are unstable for an infinite number of narrow ranges of the transverse wave number. The growth-rate is calculated explicitly for a specific example. This nonlinear instability is limited by the influence of the magnetic force which was neglected in the electrostatic approximation of the preceding sections. The total energy in the transverse oscillations is bounded by the total amount of energy in the initial electrostatic and electromagnetic perturbations.

In Section V the stationary solutions are studied. In the electrostatic approximation the cold plasma form of the Bernstein-Greene-Kruskal waves is derived. These are periodic in space and the requirement of uniqueness of the Lagrangian transformation is shown to imply the absence of trapped particles. It is also shown that the oscillations of Section II never take the form, in space, of these stationary solutions.

The transverse fields are unstable in space. The growth is again limited by the magnetic force in the longitudinal problem. Finally, an exact class of solutions is obtained for the complete problem including all magnetic force terms, and a specific example is considered in detail.

II. ONE-DIMENSIONAL ELECTROSTATIC OSCILLATIONS IN A COLD PLASMA

(a) Solution in Lagrangian Variables

Assuming that the ions form a fixed, uniform background, and that the plasma is cold, the one-dimensional Maxwell-fluid equations for the electron density, $n(x,t)$, electric field, $E(x,t)$, and electron velocity, $v(x,t)$, read in the electrostatic approximation:

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial x}(nv) = 0 \,, \tag{2.1}$$

$$\frac{\partial}{\partial t} E - 4\pi env = 0 \,, \tag{2.2}$$

$$\frac{\partial}{\partial t} v + v \frac{\partial}{\partial x} v = -\frac{e}{m} E \,, \tag{2.3}$$

and

$$\frac{\partial E}{\partial x} = -4\pi e(n-n_0) \,, \tag{2.4}$$

where n_0 is the uniform ion density. Poisson's equation may be thought of as an initial value to Eq. (2.2). By virtue of the continuity equation and Eq. (2.2), Relation (2.4) remains true for all times if true initially.

Introducing the Lagrangian variables, (x_0, τ), as new independent variables, where

$$\tau \equiv t \ ,$$

$$x_0 \equiv x - \int_0^\tau v(x_0,\tau')d\tau' \ , \tag{2.5}$$

Equations (2.1)-(2.3) may be rewritten in the new variables as:

$$\frac{\partial}{\partial\tau}\left[n(x_0,\tau)\left(1+\int_0^\tau \frac{\partial}{\partial x_0} v(x_0,\tau')d\tau'\right)\right] = 0 \ , \tag{2.6}$$

$$\frac{\partial}{\partial\tau} E(x_0,\tau) = 4\pi e n_0 v(x_0,\tau) \ , \tag{2.7}$$

and

$$\frac{\partial}{\partial\tau} v(x_0,\tau) = -\frac{e}{m} E(x_0,\tau) \ , \tag{2.8}$$

where Poisson's equation has been used in obtaining Eq. (2.7). The transformation, Eq. (2.5), has the effect of replacing the convective derivative, $\partial/\partial t + v\partial/\partial x$, by the local time derivative, $\partial/\partial\tau$. From Equations (2.7) and (2.8), $v(x_0,\tau)$ has the motion of a simple harmonic oscillator, oscillating at the plasma frequency, ω_0, i.e.,

$$\frac{\partial^2}{\partial\tau^2} v(x_0,\tau)+\omega_0^2 v(x_0,\tau) = 0; \quad \omega_0^2 = \frac{4\pi n_0 e^2}{m} \ . \tag{2.9}$$

The general solutions to the system (2.6)-(2.8) are then simply,

$$v(x_0,\tau) = V(x_0)\cos\omega_0\tau+\omega_0 X(x_0)\sin\omega_0\tau \ , \tag{2.10}$$

$$E(x_0,\tau) = \frac{m}{e}\,\omega_0 V(x_0)\sin\omega_0\tau - \frac{m}{e}\,\omega_0^2 X(x_0)\cos\omega_0\tau \ , \tag{2.11}$$

and

$$n(x_0,\tau) = \frac{n(x_0,0)}{\left[1+\frac{1}{\omega_0}\frac{\partial}{\partial x_0} V(x_0)\sin\omega_0\tau+\frac{\partial}{\partial x_0} X(x_0)(1-\cos\omega_0\tau)\right]} \ . \tag{2.12}$$

The functional dependence of V and X on x_0 is related to the initial velocity and electric field profiles, $v(x_0,0)$ and $E(x_0,0)$, through

$$V(x_0) = v(x_0,0), \quad \text{and} \quad X(x_0) = \frac{-e}{m\omega_0^2} E(x_0,0) \quad . \tag{2.13}$$

In addition, $X(x_0)$ is related to the initial density, $n(x_0,0)$, through Poisson's equation at τ=0, viz.,

$$\frac{\partial}{\partial x_0} X(x_0) = \frac{n(x_0,0)}{n_0} - 1 \quad . \tag{2.14}$$

The coordinate transformation, Eq. (2.5), may now be written as,

$$\tau = t \ ,$$
$$x = x_0 + \frac{V(x_0)}{\omega_0} \sin\omega_0\tau + X(x_0)(1-\cos\omega_0\tau) \ . \tag{2.15}$$

The transformation from Lagrangian to Eulerian coordinates, i.e., the determination of x_0 as a function of x and t from Eq. (2.15), requires an explicit specification of the initial conditions, $V(x_0)$ and $X(x_0)$, and in general entails the algebraic solution to a transcendental equation. It is evident from the solutions, (2.10)-(2.12), however, that coherent oscillations at the plasma frequency, ω_0, are maintained indefinitely over the region of initial excitation.

In addition, there is a restriction of the class of initial-value problems which may be treated by the procedure just outlined. The condition that the density be non-negative initially, requires only that

$$n(x_0,0) \geq 0 \quad . \tag{2.16}$$

However, asking that the solution for the density as given by Eq. (2.12) remain non-negative and finite for all times gives the more restrictive condition

$$n(x_0,0) > n_0/2 \quad , \tag{2.17}$$

as well as

$$\frac{1}{\omega_0}\left|\frac{\partial}{\partial x_0}V(x_0)\right| < \frac{n(x_0,0)}{n_0} \quad . \tag{2.18}$$

Relation (2.14) has been used in deriving inequalities (2.17) and (2.18). Consequently, a lower bound of $n_0/2$, not zero, is placed on the amplitude of the initial density perturbation. Also, the rate of change of $V(x_0)$ with x_0 is restricted. These do not represent physical limitations on the cold plasma initial conditions, but rather mathematical limitations on the Lagrangian formalism that has been used. Mathematically if conditions (2.17) and/or (2.18) are violated for some range(s) of x_0, the transformation from Lagrangian to Eulerian coordinates, as determined from Eq. (2.15), does not remain unique for all x and t. Physically, circumstances in which these conditions are violated for some range(s) of x_0 lead to the development of multistream flow within half the period of a plasma oscillation.[1] Considerations here, however, will be restricted to initial-value problems for which inequalities (2.17) and (2.18) are satisfied.

(b) Example with Inversion to Eulerian Coordinates

As a particular non-trivial example for which the inversion to Eulerian coordinates may be carried out explicitly, let us consider initial conditions specified by a sinusoidal perturbation in density,

$$n(x_0,0) = n_0(1+\Delta\cos kx_0), \quad |\Delta| < 1/2 \ , \tag{2.19}$$

and zero velocity,

$$V(x_0) = 0 \ . \tag{2.20}$$

The solutions (2.10)-(2.12) may then be written

$$v(x_0,\tau) = \frac{\omega_0}{k}\Delta\sin kx_0 \sin\omega_0\tau \quad , \tag{2.21}$$

$$E(x_0,\tau) = -\frac{m}{e}\omega_0^2\frac{\Delta}{k}\sin kx_0\cos\omega_0\tau \quad , \tag{2.22}$$

and

$$n(x_0,\tau) = n_0 \frac{1+\Delta\cos kx_0}{1+\Delta\cos kx_0(1-\cos\omega_0\tau)} \quad . \tag{2.23}$$

In addition, the coordinate transformation, (2.15), becomes,

$$\tau = t \quad ,$$

$$kx = kx_0 + \Omega(\tau)\sin kx_0 \ , \tag{2.24}$$

where

$$\Omega(\tau) \equiv 2\Delta\sin^2\frac{\omega_0\tau}{2} \quad . \tag{2.25}$$

The condition that $|\Delta| < 1/2$ ensures that $|\Omega(\tau)| < 1$ and that the solution, $x_0(x,t)$, to Eq. (2.24) is single-valued.

The solution may be determined numerically from Eq. (2.24) at different times, and the corresponding forms of n, v, and E, in Eulerian variables deduced from Expressions (2.21)-(2.23). For example, taking $\Delta = .45$, the density is shown as a function of x in Fig. 1, at successive quarter-periods of a plasma oscillation. The x and t dependences are periodic, with periods $2\pi/k$ and $2\pi/\omega_0$, respectively. The dense regions fill in the rare regions in the first quarter period ($\omega_0 t = \pi/2$). As time goes on, the electrostatic forces are such as to cause a "bunching" of electrons around

$$kx = (2n+1)\pi \ ; \quad n = 0,\pm1,\pm2,\ldots, \tag{2.26}$$

reaching a maximum density of $5.5n_0$ at $\omega_0 t = \pi$. The dense peaks fill in the rare regions and the system reverts to its initial state. The corresponding velocity and electric-field profiles (not presented here) exhibit a steepening of the initial wave form sinkx, without change in maximum amplitude.

We return to the problem of determining explicit analytical expressions for $n(x,t)$, $v(x,t)$, and $E(x,t)$. It is evident from Eq. (2.24) that $\sin kx_0$ and $\cos kx_0$ are periodic functions of x with period $2\pi/k$. Consequently, the density, electric field, and velocity as expressed in Eulerian coordinates, possess Fourier-series representations in x. For example, v and E each contain the factor $\sin kx_0(x,t)$, which may be written as,

$$\sin kx_0(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin nkx \ , \tag{2.27}$$

where

$$a_n(t) = \frac{k}{\pi}\int_0^{2\pi} dx \sin nkx \sin kx_0(x,t) \ . \tag{2.28}$$

The integration may be carried out using the relation between x and x_0 prescribed by Eq. (2.24), giving

$$a_n(t) = (-1)^n \frac{2}{n\Omega(t)} J_n(n\Omega(t)) \ , \tag{2.29}$$

where the J_n are Bessel functions of the first kind. The electron fluid velocity and electric field in Eulerian variables are thus given by,

$$v(x,t) = \frac{\omega_0}{k} \Delta \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\Omega(t)} J_n(n\Omega(t)) \sin nkx \sin \omega_0 t \ , \tag{2.30}$$

and

$$E(x,t) = -\frac{m}{e}\frac{\omega_0^2}{k} \Delta \sum_{n=1}^{\infty} (-1)^n \frac{2}{n\Omega(t)} J_n(n\Omega(t)) \sin nkx \cos \omega_0 t \ . \tag{2.31}$$

From Eqs. (2.4) and (2.31) we obtain for the density,

$$n(x,t) = n_0 + \frac{2n_0\Delta}{\Omega(t)} \sum_{n=1}^{\infty} (-1)^n J_n(n\Omega(t)) \cos nkx \cos \omega_0 t \ . \tag{2.32}$$

Expressions (2.30)-(2.32) are thus exact solutions to the initial-value problem (2.19) and (2.20), and demonstrate quite explicitly the distortion of wave forms as manifested through the generation of higher harmonic dependence on kx. In addition, the form of Eqs. (2.30)-(2.32) is useful to provide a convergent series representation of n, v and E in powers of the amplitude, Δ, of the initial density perturbation. The reader will also have noted that the method of inversion used above is not limited to the case in which $n(x_0,0)$ has a simple sinusoidal dependence on x_0, but may be generalized to treat any initial density perturbation which itself has a Fourier series representation in x_0.

In addition, we remind the reader that the preceding analysis determines the solution to the Vlasov equation with self-consistent electric field, and initial value for the distribution function of the form,

$$f(x,\nu,0) = n_0(1+\Delta\cos kx)\delta(\nu), \quad |\Delta| < 1/2 \ . \tag{2.33}$$

The solution for all times may be written in the form

$$f(x,\nu,t) = n(x,t)\delta(\nu-v(x,t)) \ , \tag{2.34}$$

where $n(x,t)$ and $v(x,t)$ are given by Eqs. (2.30) and (2.32). The plasma which is initially cold, remains cold for all times in the sense that no random motion relative to $v(x,t)$ develops. For more general initial conditions for the density, $n(x,0)$, and mean velocity $v(x,0)$, the solution for $f(x,\nu,t)$ in the cold plasma problem may still be written in the form given by Eq. (2.34), provided inequalities (2.17) and (2.18) are satisfied and consequently multistream flow does not develop.

III. GENERALIZATIONS OF THE ELECTROSTATIC PROBLEM

(a) Nonlinear zero-temperature Bernstein Modes[3]

The analysis of Section II may be generalized to describe the nonlinear behavior of large-amplitude electrostatic oscillations perpendicular to a uniform, static magnetic field, $\underset{\sim}{B}_0$. With unit Cartesian vectors $(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3)$, we take $\underset{\sim}{B}_0$ along $\hat{\underset{\sim}{e}}_3$, and as before consider spatial variations in the x-direction $(\hat{\underset{\sim}{e}}_1)$. The electron velocity, $\underset{\sim}{v}(x,t)$, may be written as,

$$\underset{\sim}{v}(x,t) = v(x,t)\hat{\underset{\sim}{e}}_1+v_2(x,t)\hat{\underset{\sim}{e}}_2+v_3(x,t)\hat{\underset{\sim}{e}}_3 \ , \tag{3.1}$$

and the electric field, which is along the direction of spatial variations in the electrostatic approximation, becomes,

$$\underset{\sim}{E}(x,t) = E(x,t)\hat{\underset{\sim}{e}}_1 \ . \tag{3.2}$$

The plasma is again assumed cold. Equations (2.1) and (2.2) describing the evolution of $E(x,t)$ and $n(x,t)$, remain unchanged. However, the electron velocity in the $\hat{\underset{\sim}{e}}_1$-direction now obeys

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = -\,\frac{e}{m}\,E-\Omega_0 v_2; \quad \Omega_0 \equiv \frac{eB_0}{mc} \ , \tag{3.3}$$

and the two additional degrees of motion satisfy,

$$\frac{\partial}{\partial t}\,v_2 + v\,\frac{\partial}{\partial x}\,v_2 = \Omega_0 v \ , \tag{3.4}$$

and

$$\frac{\partial}{\partial t}\,v_3 + v\,\frac{\partial}{\partial x}\,v_3 = 0 \ . \tag{3.5}$$

Defining Lagrangian coordinates following the x-motion by Eq. (2.5), Eqs. (3.3)-(3.5) read in the new variables,

$$\frac{\partial}{\partial \tau}\,v(x_0,\tau) = -\,\frac{e}{m}\,E(x_0,\tau)-\Omega_0 v_2(x_0,\tau) \ , \tag{3.6}$$

$$\frac{\partial}{\partial \tau}\,v_2(x_0,\tau) = \Omega_0 v(x_0,\tau) \ , \tag{3.7}$$

and

$$\frac{\partial}{\partial \tau}\,v_3(x_0,\tau) = 0 \ . \tag{3.8}$$

The system of equations to be solved in Lagrangian variables now consists of (2.6), (2.7) and (3.6)-(3.8). From Eqs. (2.7), (3.6) and (3.7), $v(x_0,\tau)$ is seen to have the motion of a simple harmonic oscillator, oscillating at the upper hybrid frequency, ω_{UH}, i.e.,

$$\frac{\partial^2}{\partial \tau^2}\,v(x_0,\tau)+\omega_{UH}^2 v(x_0,\tau) = 0; \quad \omega_{UH}^2 = \omega_0{}^2+\Omega_0{}^2 \ . \tag{3.9}$$

Consequently, the only modifications of the solutions (2.10) and (2.12) for $v(x_0,\tau)$ and $n(x_0,\tau)$, and the coordinate transformation (2.15), is that ω_0 is replaced by ω_{UH}. The quantities, $v_2(x_0,\tau)$, $E(x_0,\tau)$ and $v_3(x_0,\tau)$, as determined from Eqs. (3.6)-(3.8), are then given by

$$v_2(x_0,\tau) = v_2(x_0,0) + \frac{\Omega_0}{\omega_{UH}} \{V(x_0)\sin\omega_{UH}\tau + \omega_{UH}X(x_0)(1-\cos\omega_{UH}\tau)\} \quad , \tag{3.10}$$

$$E(x_0,\tau) = \frac{m}{e}\frac{\omega_0^2}{\omega_{UH}} V(x_0)\sin\omega_{UH}\tau - \frac{m}{e}\omega_0^2 X(x_0)\cos\omega_{UH}\tau - \frac{m}{e}(\Omega_0^2 X(x_0)+\Omega_0 v_2(x_0,0)) \; , \tag{3.11}$$

and

$$v_3(x_0,\tau) = v_3(x_0,0) \; , \tag{3.12}$$

thus giving a complete description of the problem in Lagrangian variables. The explicit behavior in Eulerian variables depends on the details of the initial conditions chosen for the problem. However, it is clear that coherent oscillations at the upper hybrid frequency are maintained for all time in the region of initial excitation. In terms of the initial conditions, $V(x_0)$, $v_2(x_0,0)$ and $n(x_0,0)$, the conditions that the transformation from Lagrangian to Eulerian coordinates be unique and that multistream flow does not develop, now become,

$$\frac{n(x_0,0)}{n_0} - \frac{\Omega_0}{\omega_0^2}\frac{\partial}{\partial x_0} v_2(x_0,0) > \frac{\omega_0^2-\Omega_0^2}{2\omega_0^2} \; , \tag{3.13}$$

and

$$\frac{1}{\omega_{UH}}\left|\frac{\partial}{\partial x_0} V(x_0)\right| < \frac{\Omega_0^2}{\omega_{UH}^2} + \frac{\omega_0^2}{\omega_{UH}^2}\frac{n(x_0,0)}{n_0} - \frac{\Omega_0}{\omega_{UH}^2}\frac{\partial}{\partial x_0} v_2(x_0,0) \; . \tag{3.14}$$

Conditions (3.13) and (3.14) reduce to the inequalities (2.16) and (2.17) for $\Omega_0 \to 0$, as they should. If there is no initial shear in the electron velocity $(\partial v_2(x_0,0)/\partial x_0 = 0)$, conditions (3.13) and (3.14) become less and less restrictive with increasing magnetic field strength.

(b) Thermal Effects

Modifications due to dispersion resulting from finite plasma temperature may be described by including the force due to pressure variations in Eq. (2.3), i.e.,

$$\frac{\partial v}{\partial t} + v \frac{\partial}{\partial x} v = - \frac{e}{m} E - \frac{1}{nm} \frac{\partial}{\partial x} P \ . \tag{3.15}$$

The evolution of the electron pressure, P, as determined from the appropriate moments of the Vlasov equation, is given by

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x}\right)\left(\frac{P}{n^3}\right) = 0 \ , \tag{3.16}$$

in circumstances where the electron heat flow may be neglected.[4] This corresponds to the approximation, $(\omega_0/k) \gg v_{TH}$, where $1/k$ is the (typical) length scale of the disturbance being studied, and v_{TH} is the electron thermal speed. In Lagrangian variables defined by Eq. (2.5), Eqs. (2.6) and (2.7) remain unchanged with inclusion of pressure effects; however, the equation for $v(x_0,\tau)$ is now given by

$$\frac{\partial}{\partial \tau} v(x_0,\tau) = - \frac{e}{m} E(x_0,\tau) - \frac{1}{n(x_0,0)m} \frac{\partial}{\partial x_0} P(x_0,\tau) \ , \tag{3.17}$$

where

$$P(x_0,\tau) = \frac{n(x_0,\tau)^3}{n(x_0,0)^3} P(x_0,0) \ . \tag{3.18}$$

Differentiating Eq. (3.17) with respect to τ, we then have that

$$\frac{\partial^2}{\partial \tau^2} v(x_0,\tau) + \omega_0{}^2 v(x_0,\tau) = \frac{3}{n(x_0,0)m} \frac{\partial}{\partial x_0} \left\{ \frac{P(x_0,0) \dfrac{\partial}{\partial x_0} v(x_0,\tau)}{\left(1 + \int_0^\tau \dfrac{\partial}{\partial x_0} v(x_0,\tau')d\tau'\right)^4} \right\} \ . \tag{3.19}$$

The solution to Eq. (3.19) is not mathematically tractable except within some additional approximation scheme; we discuss only one of these here. It should be noted that in an order of magnitude estimate the thermal effects are smaller by a factor $k^2 v_{TH}^2/\omega_0^2$ relative to the other terms in Eq. (3.19). For purposes of a qualitative description, the model we adopt is one in which the final term in Eq. (3.19) is approximated by its linearized version. In particular, for the case of an (initially) uniform pressure P_0, Eq. (3.19) becomes

$$\frac{\partial^2}{\partial\tau^2} v(x_0,\tau)+\omega_0^2 v(x_0,\tau) = 3\omega_0^2\lambda_D^2 \frac{\partial^2}{\partial x_0^2} v(x_0,\tau) \tag{3.20}$$

where

$$\lambda_D^2 \equiv \frac{P_0}{n_0 m\omega_0^2} \quad . \tag{3.21}$$

Strictly speaking, Eq. (3.20) is applicable only in a small amplitude analysis. However, it may be expected to give qualitatively correct behavior even for moderately large (initial) amplitudes provided steep spatial grandients do not develop in the course of time (as was the case for the initial conditions chosen for Fig. 1, Section II). For initial values of the form of a single monochromatic wave of wave number k, the analysis goes through as in Section II-(b), the essential modification being a slight frequency shift from ω_0 to $\omega_0(1+3k^2\lambda_D^2)^{1/2}$. However, for initial conditions which are absolutely integrable, the long time solution to Eq. (3.20) has quite different behavior than for the case of a cold plasma. The analysis of Section II indicated that coherent plasma oscillations are maintained indefinitely in the region of initial excitation, whereas, in relation to Eq. (3.20) the inclusion of thermal effects leads to a dispersion of the disturbance throughout the plasma. The solution to Eq. (3.20) may be written in terms of its Fourier integral representation as

$$v(x_0,\tau) = \int dk_0 e^{ik_0x_0}\{V(k_0)\cos\omega(k_0)\tau+\omega(k_0)X(k_0)\sin\omega(k_0)\tau\} \tag{3.22}$$

where

$$\omega(k_0) = \omega_0(1+3k_0^2\lambda_D^2)^{1/2} \ . \tag{3.23}$$

The quantity, $V(k_0)$, is the Fourier transform of the initial velocity perturbation, $v(x_0,0)$, and $X(k_0)$ is related to the transform of the initial electric field by

$$\omega^2(k_0)X(k_0) = -\frac{e}{m}E(k_0,0) \ . \tag{3.24}$$

A straight forward stationary phase analysis of Eq. (3.22) indicates that the oscillatory terms phase mix to zero as $(1/\tau^{1/2})$ for large τ.[5] For example, the first term decays as

$$\left(\frac{\pi}{3\omega_0\tau\lambda_D^2}\right)^{1/2} V(k_0=0) \times (\text{oscillation at frequency } \omega_0), \tag{3.25}$$

for large τ. Estimating $V(k_0=0) \cong \overline{V}\Delta$ where $\overline{V}$ and Δ are the characteristic magnitude and (spatial) width, respectively, of the initial disturbance $v(x_0,0)$, we see that $v(x_0,\tau)$ decays to a negligible level compared to $\overline{V}$ in a time t, where

$$\omega_0 t \gg \Delta^2/\lambda_D^2 \ . \tag{3.26}$$

It should be noted that the time integrations over $v(x_0,\tau)$ involved in the transformation (2.5), and the solution for $n(x_0,\tau)$ as determined from Eq. (2.6), have non oscillatory portions which do not phase mix to zero. The time asymptotic behavior is conveniently written in terms of $X(x_0)$ and may be summarized as follows:

$$v(x_0,\tau\to\infty) \to 0 \ , \tag{3.27}$$

$$n(x_0,\tau\to\infty) \to n_0 - 3n_0\lambda_D^2 \frac{\partial^3 X(x_0)}{\partial x_0^3} \ , \tag{3.28}$$

and

$$E(x_0,\tau\to\infty) \to 12\pi n_0 e\lambda_D^2 \frac{\partial^2 X(x_0)}{\partial x_0^2} \ , \tag{3.29}$$

where $x_0(x,\tau\to\infty)$ is determined from

$$x = x_0 + X(x_0) \ . \tag{3.30}$$

That is to say, a low level, stationary electric field remains, together with its associated density profile. This represents a balance between electrostatic force and the (long time) force due to pressure variations in the model that has been used.

(c) Dissipation due to Collisional Drag

The effect of dissipation in the nonlinear analysis of Section II may be simulated by including a collisional drag term, $-\nu v$, on the right-hand side of the equation of motion, Eq. (2.3). The only modification of the system (2.5)-(2.8) occurs for Eq. (2.8) which now reads,

$$\frac{\partial}{\partial \tau} v(x_0,\tau) = -\frac{e}{m} E(x_0,\tau) - \nu v(x_0,\tau) \ . \tag{3.31}$$

The velocity in Lagrangian variables then satisfies

$$\frac{\partial^2}{\partial \tau^2} v(x_0,\tau) + \nu \frac{\partial}{\partial \tau} v(x_0,\tau) + \omega_0^{\,2} v(x_0,\tau) = 0 \ , \tag{3.32}$$

where ν has been assumed constant. The motion thus exhibits damped oscillations, with a damping factor $\exp(-\frac{\nu}{2}\tau)$, and oscillation frequency $(\omega_0^{\,2}-\nu^2/4)^{1/2}$, where $\omega_0 > \nu/2$ by assumption. Consequently, $v(x_0,\tau)$ tends to zero for large τ, as does the electric field, $E(x_0,\tau)$. Similarly, the density may be shown to damp to the value of the uniform background density, n_0. This asymptotic time behavior of course persists in the Eulerian frame, and is valid for any initial conditions that do not lead to multistream flow. For the case,

$$\nu \ll \omega_0 \ , \tag{3.33}$$

this restriction is still given (approximately) by inequalities (2.17) and (2.18).

IV. ELECTROMAGNETIC OSCILLATIONS

The complete set of cold plasma equations in situations where all gradients are in the x-direction can be conveniently split up into three subsets.

Subset A.

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv_x) = 0 \quad , \tag{4.1}$$

$$\frac{\partial E_x}{\partial x} = -4\pi e(n-n_0) \quad , \tag{4.2}$$

$$\frac{\partial E_x}{\partial t} = 4\pi env_x \quad , \tag{4.3}$$

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} = -\frac{e}{m}(E_x + \frac{1}{c}v_yB_z - \frac{1}{c}v_zB_y) \quad . \tag{4.4}$$

This subset describes the longitudinal oscillations treated in the preceding sections if the contributions to the Lorentz force in Eq. (4.4) due to self-consistent magnetic field are neglected. This neglect is apparently justified if the fluid velocities are much smaller than the velocity of light and if the fields B_y, B_z arising from electromagnetic oscillations are not much larger than the electrostatic field E_x.

Subset B.

$$\frac{\partial E_y}{\partial t} = 4\pi env_y - c\frac{\partial B_z}{\partial x} \quad , \tag{4.5}$$

$$\frac{\partial B_z}{\partial t} = -c\frac{\partial E_y}{\partial x} \quad , \tag{4.6}$$

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} = -\frac{e}{m}(E_y - \frac{1}{c}v_xB_z) \quad . \tag{4.7}$$

This set describes transverse oscillations. We have put B_x (satisfying $\partial B_x/\partial x = 0$) equal to zero.

Subset C.

$$\frac{\partial E_z}{\partial t} = 4\pi e n v_z + c\,\frac{\partial B_y}{\partial x} \quad , \tag{4.8}$$

$$\frac{\partial B_y}{\partial t} = c\,\frac{\partial E_z}{\partial x} \quad , \tag{4.9}$$

$$\frac{\partial v_z}{\partial t} + v_x\,\frac{\partial v_z}{\partial x} = -\frac{e}{m}\left(E_z + \frac{1}{c}\,v_x B_y\right) \quad . \tag{4.10}$$

The system (4.8)-(4.10) describes the additional degree of freedom for transverse oscillations and is identical to subset B with the interchange of y and z, and the replacement of c by -c.

We now treat the case without external magnetic field, and in addition neglect the magnetic force in Eq. (4.4). The solutions of subset A are then known from Section II. Consequently, the quantities n(x,t) and $v_x(x,t)$ enter the equations (4.5)-(4.10) as known functions. We solve the equations of subset B keeping in mind that the solutions of Eqs. (4.8)-(4.10) may be written down by analogy, and introduce the vector potential A_y related to the fields by

$$E_y = -\frac{1}{c}\,\frac{\partial A_y}{\partial t} \,, \quad B_z = \frac{\partial A_y}{\partial x} \,. \tag{4.11}$$

This satisfies Eq. (4.6), and (4.7) reduces to

$$\left(\frac{\partial}{\partial t} + v_x\,\frac{\partial}{\partial x}\right)\left(v_y - \frac{e}{mc}\,A_y\right) = 0 \,, \tag{4.12}$$

expressing conservation of canonical momentum in a reference system moving with v_x. Equation (4.5) becomes

$$\frac{\partial^2 A_y}{\partial x^2} - \frac{1}{c^2}\,\frac{\partial^2 A_y}{\partial t^2} = \frac{4\pi e}{c}\,n v_y \quad . \tag{4.13}$$

From (4.12) we obtain v_y in terms of A_y:

$$v_y = \frac{e}{mc} A_y + v_y(x_0, t=0) - \frac{e}{mc} A_y(x_0, t=0) \ , \tag{4.14}$$

where x_0 is the solution of

$$x_0 = x - \int_0^t v_x(x_0, t')dt' \tag{4.15}$$

Substituting (4.14) into (4.13) it follows that

$$\frac{\partial^2 A_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A_y}{\partial t^2} - \frac{\omega_0^{\ 2}}{c^2} NA_y = F(x,t) \ , \tag{4.16}$$

with

$$\omega_0^{\ 2} = \frac{4\pi e^2 n_0}{m} \ , \quad N = \frac{n(x,t)}{n_0} \ ,$$

and

$$F(x,t) = \frac{4\pi en(x,t)}{c} [v_y(x_0, t=0) - \frac{e}{mc} A_y(x_0, t=0)] \ . \tag{4.17}$$

The function $N(x,t)$ is periodic in t. Therefore Eq. (4.16) suggests the possibility of parametric resonance. In Section II expressions for $N(x,t)$ and $v_x(x,t)$ were obtained for the case of an initially sinusoidal density perturbation and $v_x(x,t=0) = 0$. We use these results in order to investigate the properties of Eq. (4.16) explicitly. However, generalizations to other initial conditions for the longitudinal problem are straightforward.

We denote the characteristic wave numbers of the longitudinal and transverse oscillations by k_0 and k respectively and the characteristic frequency of the transverse oscillations by ω. Since $N(x,t)$ and $F(x,t)$ in Eq. (4.16) vary in space and time with characteristic wave number k_0 and frequency ω_0, considerations will be restricted to solutions, A_y, with $k \gtrsim k_0$ and $\omega \gtrsim \omega_0$. We discuss four different ordering schemes.

1) $\underline{\omega \sim \omega_0, \ kc \gg \omega_0}$

Eq. (4.16) reduces to

$$\frac{\partial^2 A_y}{\partial x^2} = F(x,t) \ , \tag{4.18}$$

describing the direct generation of the magnetic field $B_z = \partial A_y/\partial x$ by the electric current density, $cF(x,t)$.

2) $\underline{\omega \sim kc \gg \omega_0}$

Eq. (4.16) reduces to

$$\frac{\partial^2 A_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A_y}{\partial t^2} = F(x,t) \quad . \tag{4.19}$$

The solutions of the homogeneous part of the equation are plane electromagnetic waves. The inhomogeneous equation (4.19) is easily solved by transformation to the new variables, $x_\pm = x \pm ct$. The general solution of Eq. (4.19) in terms of the initial values of the fields is given by

$$\begin{aligned} E_y &= \frac{1}{2}\,[E_y(x-ct,t=0)+E_y(x+ct,t=0)+B_z(x-ct,t=0) \\ &\quad - B_z(x+ct,t=0)] + \frac{1}{2}\int_{x-ct}^{x+ct} dx' F(x',t-\frac{|x'-x|}{c}) \ , \\ B_z &= \frac{1}{2}\,[B_z(x-ct,t=0)+B_z(x+ct,t=0)+E_y(x-ct,t=0) \\ &\quad - E_y(x+ct,t=0)] + \frac{1}{2}\int_{x-ct}^{x+ct} dx'\,\mathrm{sgn}(x-x')F(x',t-\frac{|x'-x|}{c}) \ , \end{aligned} \tag{4.20}$$

where the appearance of retardation is clear. The difference from the analogous three-dimensional expressions for the particular solution of the inhomogeneous wave equation should be noted. Equation (4.20) describes the generation of high frequency short wavelength transverse fields.

3) $\underline{k \gg k_0,\ \omega \sim \omega_0 \sim kc}$

This limiting case has the effect of introducing two space scales in Eq. (4.16). Reserving x for the fast variations in space, and introducing $\xi = k_0 x$ for the slow variations, we see that $\partial/\partial x$ is replaced by $\partial/\partial x + k_0\,\partial/\partial\xi$. In lowest order, Eq. (4.16) then becomes

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_0^{\ 2}}{c^2} N(\xi,t)\right] A_y(x,\xi,t) = F(x,\xi,t) \quad . \tag{4.21}$$

We start by studying the solutions, $A_y^{\ H}$, of the homogeneous part of Eq. (4.21). Introducing the Fourier transform,

$$A_k(\xi,t) = \int_{-\infty}^{+\infty} A_y^{\ H}(x,\xi,t)e^{-ikx}dx \ , \tag{4.22}$$

it follows that,

$$\frac{\partial^2 A_k}{\partial t^2} + \{k^2c^2 + \omega_0^{\ 2}N(\xi,t)\}A_k = 0 \ . \tag{4.23}$$

Since $N(\xi,t)$ is periodic in t this is a Hill equation. In order to discuss Equation (4.23) and make use of well known results of the theory of Mathieu and Hill equations,[6,7] we write $N(\xi,t)$ as a double Fourier series for the case of an initially sinusoidal density perturbation with $v_x(x,t=0) = 0$, (the example discussed in detail in Section II), i.e.,

$$N(\xi,t) = \sum_{\ell=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} N_{\ell,m} \exp[-i\ell T + im\xi] \quad , \tag{4.24}$$

where $T = \omega_0 t$. Then with $\xi_0 = k_0 x_0$ and Eqs. (2.23)-(2.25) we find

$$N_{\ell,m} = \frac{1}{(2\pi)^2}\int_0^{2\pi} dT \int_0^{2\pi} d\xi_0 N(\xi_0,T) \frac{\partial\xi}{\partial\xi_0} \exp[i\ell T - im\xi(\xi_0,T)]$$

$$= \frac{1}{(2\pi)^2}\int_0^{2\pi} dT \int_0^{2\pi} d\xi_0 (1+\Delta\cos\xi_0)\exp[i\ell T - im\xi_0 - im\Delta(1-\cos T)\sin\xi_0] \ .$$

Using standard trigonometric identities and series expansions with Bessel functions the integrals can be evaluated with the result

$$N_{\ell,m} = \sum_{q=-\infty}^{+\infty} \frac{\ell+2q}{m} J_{\ell+2q-m}(m\Delta)J_q(m\Delta/2)J_{\ell+q}(m\Delta/2) \quad . \tag{4.25}$$

The following properties are readily established:

$$N_{0,0} = 1, \quad N_{\ell,0} = 0 \text{ for } \ell \neq 0, \quad N_{\ell,m} = N_{-\ell,m} = N_{\ell,-m} = N_{-\ell,-m} \ .$$

Therefore

$$N(\xi,t) = 1+2Y(\xi)+4\sum_{\ell=1}^{\infty} N_\ell(\xi)\cos\ell T \quad , \tag{4.26}$$

with

$$Y(\xi) = \sum_{m=1}^{\infty} N_{0,m}\cos m\xi \quad , \tag{4.27}$$

and

$$N_\ell(\xi) = \sum_{m=1}^{\infty} N_{\ell,m}\cos m\xi \quad . \tag{4.28}$$

If $\Delta \ll 1$ we may expand in powers of Δ and obtain after considerable algebra

$$\begin{aligned} N = {} & 1+\Delta\cos\xi\cos T+\Delta^2\cos 2\xi[\tfrac{1}{2} -\cos T+ \tfrac{1}{2}\cos 2T] \\ & + \Delta^3[\cos\xi\{\tfrac{1}{8} - \tfrac{7}{32}\cos T+ \tfrac{1}{8}\cos 2T- \tfrac{1}{32}\cos 3T\} \\ & + \cos 3\xi\{- \tfrac{9}{8} + \tfrac{63}{32}\cos T- \tfrac{9}{8}\cos 2T+ \tfrac{9}{32}\cos 3T\}]+O(\Delta^4) \quad . \end{aligned} \tag{4.29}$$

Equation (4.23) may be written in the standard form[6,7]

$$\frac{\partial^2 A_k}{\partial\tau^2} + \left[\theta_0+2\sum_{\ell=1}^{\infty} \theta_{2\ell}\cos 2\ell\tau\right] A_k = 0 \quad , \tag{4.30}$$

where $\tau = T/2$. The $\theta_{2\ell}$ follow immediately from Eqs. (4.25)-(4.28). For small Δ we get from Eq. (4.29)

$$\begin{aligned} \theta_0 &= 4\left(1+ \frac{k^2c^2}{\omega_0^2}\right) + \frac{1}{2}[\Delta^3\cos\xi+4\Delta^2\cos 2\xi-9\Delta^3\cos 3\xi]+O(\Delta^4) \\ \theta_2 &= 2\Delta[(1- \tfrac{7}{32}\Delta^2)\cos\xi-\Delta\cos 2\xi+ \tfrac{63}{32}\Delta^2\cos 3\xi]+O(\Delta^4) \\ \theta_4 &= \Delta^2[\tfrac{\Delta}{4}\cos\xi+\cos 2\xi- \tfrac{9}{4}\Delta\cos 3\xi]+O(\Delta^4) \\ \theta_6 &= \frac{\Delta^3}{16}(-\cos\xi+9\cos 3\xi)+O(\Delta^4) \\ \theta_8 &= O(\Delta^4) \quad . \end{aligned} \tag{4.31}$$

It is well known[6,7] that the solutions of the Hill equation, Eq. (4.30), have the form

$$A_k = a\ \exp(\mu\tau)\phi(\tau)+b\ \exp(-\mu\tau)\phi(-\tau)\ , \tag{4.32}$$

where a and b are integration constants, and $\phi(\tau)$ is a periodic function. Depending on the coefficients $\theta_{2\ell}$, μ is either purely imaginary implying stability, or of the form, $ri+\mu_R$, where r is an integer or zero and μ_R is real, implying instability. It is the latter possibility which is of special interest here.

The quantity μ is given by[6]

$$\cosh\pi\mu = 1-2S(0)\sin^2\frac{\pi}{2}\ \sqrt{\theta_0}\ , \tag{4.33}$$

where S(0) is the infinite determinant,

$$S(0) = \begin{vmatrix} & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & 1 & \frac{\theta_2}{\theta_0-16} & \frac{\theta_4}{\theta_0-16} & \frac{\theta_6}{\theta_0-16} & \frac{\theta_8}{\theta_0-16} & \cdot \\ \cdot & \frac{\theta_2}{\theta_0-4} & 1 & \frac{\theta_2}{\theta_0-4} & \frac{\theta_4}{\theta_0-4} & \frac{\theta_6}{\theta_0-4} & \cdot \\ \cdot & \frac{\theta_4}{\theta_0} & \frac{\theta_2}{\theta_0} & 1 & \frac{\theta_2}{\theta_0} & \frac{\theta_4}{\theta_0} & \cdot \\ \cdot & \frac{\theta_6}{\theta_0-4} & \frac{\theta_4}{\theta_0-4} & \frac{\theta_2}{\theta_0-4} & 1 & \frac{\theta_2}{\theta_0-4} & \cdot \\ \cdot & \frac{\theta_8}{\theta_0-16} & \frac{\theta_6}{\theta_0-16} & \frac{\theta_4}{\theta_0-16} & \frac{\theta_2}{\theta_0-16} & 1 & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \end{vmatrix} \tag{4.34}$$

It has been assumed that $\theta_0 \neq 4r^2$, $r = 0,\pm1$ etc.

If all terms of $O(\Delta^2)$ are neglected, then Eq. (4.30) is a Mathieu equation with stability diagram[6,7] as given in Fig. 2. We see from Eq. (4.31) that $\theta_0 > 4$ and the first unstable region is near $\theta_0 = 9$. Therefore we consider Eq. (4.30) with

$$\theta_0 = 4\left(1+\frac{k^2c^2}{\omega_0^2}\right) \cong 9, \quad \theta_2 = 2\Delta\cos\xi, \quad \theta_{2\ell} = 0 \text{ for } \ell > 1\ . \tag{4.35}$$

In this case we use the results of Ref. 6, page 17, describing the boundary curves of the stable regions in Fig. 2 in order to find the region of instability. Unstable waves develop for k in the range

$$\frac{5}{4} + \frac{1}{16}\Delta^2\cos^2\xi - \frac{1}{32}\Delta^3|\cos\xi|^3 < \frac{k^2c^2}{\omega_0^2} < \frac{5}{4} + \frac{1}{16}\Delta^2\cos^2\xi + \frac{1}{32}\Delta^3|\cos\xi|^3 \ . \tag{4.36}$$

The width of this range is clearly of order Δ^3. The maximal growth rate follows from Ref. 6, page 101, and is given by

$$\mu_{Rmax} = \frac{1}{48}\Delta^3|\cos\xi|^3 \ . \tag{4.37}$$

The growth rate takes this maximal value when k^2c^2/ω_0^2 is approximately in the middle of the interval (4.36). On both sides of the middle μ_R decreases until it vanishes at the boundaries of the interval. In a similar way results may be obtained near $\theta_0 = 16,25...$ with growth rates and unstable k ranges of order Δ^4,Δ^5 etc.

If Δ is not small (but still $\Delta < 1/2$) it can be shown that the unstable regions and the growth rates are much larger. This is suggested by Fig. 2. This stability diagram, however, is no longer valid because $\theta_{2\ell}$, $\ell > 1$ will also play a role. It should also be realized that since $v_x \sim \Delta\omega_0/k_0 \sim \Delta ck/k_0$ the present non-relativistic treatment is only justified if

$$\Delta \ll \frac{k_0}{k} \ll 1 \ . \tag{4.38}$$

Therefore larger values of Δ together with $k_0 \ll k$ require a relativistic treatment or restriction to spatial regions for $k_0x \sim n\pi$ (n is an integer) since there v_x is small according to Eq. (2.21). It must also be recognized that Eq. (4.23) breaks down after a sufficiently long time. This can be seen as follows. If we take a derivative with respect to x in Eq. (4.32), a factor

t appears since μ depends on k_0x (see Eq. (4.37)). Since such terms were neglected in writing Eq. (4.23) we have the requirement

$$\omega_0 t\Delta^3 k_0 \ll k \quad . \tag{4.39}$$

Consequently the explicit calculations are only valid for a restricted time; however, considerable growth is possible in Eq. (4.32) since the growth rate is of order $\omega_0\Delta^3$ and $k_0 \ll k$. Secularities might be avoided by means of a generalized W.K.B.-method and it is therefore not to be expected that the restriction (4.39) represents a serious limitation of the theory.

Questions may be raised regarding the energy source of the instability. It is clear from total conservation of energy that the amount of energy in the unstable transverse oscillations can never exceed the energy content of the initial longitudinal and transverse perturbations. It is of some interest to examine the energy balance for the longitudinal and transverse oscillations separately. It can be easily shown that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\, nmv_x^{\,2} + \frac{E_x^{\,2}}{8\pi}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\, nmv_x^{\,3}\right) = -K \ , \tag{4.40}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\, nmv_y^{\,2} + \frac{E_y^{\,2}+B_z^{\,2}}{8\pi}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\, nmv_y^{\,2}v_x + \frac{c}{4\pi}\, E_yB_z\right) = +K \ , \tag{4.41}$$

where

$$K = \frac{v_x}{4\pi c}\,\frac{\partial}{\partial t}\,(E_yB_z) + v_x\,\frac{\partial}{\partial x}\,\frac{E_y^{\,2}+B_z^{\,2}}{8\pi} \quad . \tag{4.42}$$

The right hand side of Eq. (4.40) is due to the magnetic force $-e/mc\ v_yB_z$ in Eq. (4.4) which was neglected in preceding calculations. The quantity K in Eq. (4.41), however, was <u>not</u> neglected in the treatment of the transverse oscillations. It is this quantity which makes the transfer of energy from the longitudinal into the transverse oscillations possible. If we start initially with $E_y^{\,2}+B_z^{\,2} \ll E_x^{\,2}$ then ultimately $E_y^{\,2}+B_z^{\,2}$ is of the same order of magnitude as E_x^2. Comparing the magnetic force with the electrostatic force in Eq. (4.4) it is easily seen that

$$\frac{v_y B_z}{cE_x} \sim \frac{k\Delta}{k_0} \frac{E_y^2}{E_x^2} \quad . \tag{4.43}$$

Therefore in view of Eq. (4.38) the magnetic force is still relatively small when the energy of the transverse oscillations has reached its ultimate level. Nevertheless it is this magnetic force that is responsible for turning off the instability.

Thus far, only the solution to the homogeneous part of Eq. (4.21) has been discussed. This corresponds to initial conditions with zero canonical momentum in the y-direction as is seen from Eq. (4.17). In general, if $F(x,0) \neq 0$, we must add a particular solution of the complete inhomogeneous equation. For instance instead of Eq. (4.23) we have

$$\frac{\partial^2 A_k}{\partial t^2} + \{k^2c^2+\omega_0^2 N(\xi,t)\}A_k = F_k(\xi,t) \quad , \tag{4.44}$$

where $A_k(\xi,t)$ and $F_k(\xi,t)$ are the Fourier transforms of $A(x,\xi,t)$ and $F(x,\xi,t)$ respectively. A solution of Eq. (4.44) is

$$A_k = \tilde{A}_k(\xi,t)\left[1+\int^t \frac{dt'}{\tilde{A}_k^2(\xi,t')}\int^{t'} F_k(\xi,t'')\tilde{A}_k(\xi,t'')dt''\right] \quad , \tag{4.45}$$

where $\tilde{A}_k(\xi,t)$ is a solution of the homogeneous part of Eq. (4.44). The zeros of $\tilde{A}_k(\xi,t')$ in Eq. (4.45) can be accommodated by suitable choice of integration contour in the complex t'-plane.

4) $\underline{k \sim k_0,\ kc \ll \omega_0}$

In this case Eq. (4.16) reduces to

$$\frac{\partial^2 A_y}{\partial t^2} + \omega_0^2 N(x,t)A_y = c^2F(x,t) \quad , \tag{4.46}$$

and we recover Eq. (4.30) and Eq. (4.31) where now $\theta_0 = 4+O(\Delta^2)$. Instability seems possible again since $\theta_0=4$ is a point where an unstable region extends to the θ_0-axis in the stability diagram, Fig. 2. However, we remind the reader that the width of the unstable region is of order Δ^2 and the neglect of k^2c^2/ω_0^2 in θ_0 as given by Eq. (4.31) is only correct if $kc < \omega_0\Delta$. Therefore with $k \sim k_0$ we find in general that $v_x \sim \omega_0\Delta/k_0 > c$ which is inconsistent. Consequently, the unstable solution is possible only for $k_0x \sim n\pi$ (n being an integer) where v_x is small according to Eq. (2.21).

V. STATIONARY SOLUTIONS

We now investigate time-independent solutions of the Eqs. (4.1)-(4.7). Again solutions of Eqs. (4.8)-(4.10) are quite analogous to those of Eqs. (4.5)-(4.7). We will treat the purely electrostatic problem leading to cold plasma version of the Bernstein-Greene-Kruskal (B.G.K.) waves,[8] the electromagnetic problem using the electrostatic results, and a special case of the complete problem including the magnetic force in Eq. (4.4).

(a) Electrostatic Problem

Allowing for a constant velocity, v_0, of the uniform background, the relevant equations are

$$nv_x = n_0v_0 \ , \tag{5.1}$$

$$v_x \frac{dv_x}{dx} = - \frac{e}{m} E_x \ , \tag{5.2}$$

$$\frac{dE_x}{dx} = -4\pi e(n-n_0) \ . \tag{5.3}$$

We introduce the Lagrangian coordinate η by

$$x = \int_0^\eta v_x(\eta')d\eta' \ , \tag{5.4}$$

implying

$$v_x \frac{d}{dx} = \frac{d}{d\eta} \; .$$

The equation for v_x becomes

$$\frac{d^2 v_x}{d\eta^2} + \omega_0^2 v_x = \omega_0^2 v_0 \; . \tag{5.5}$$

Consequently

$$v_x = v_0 + a\cos\omega_0\eta + b\sin\omega_0\eta \; , \tag{5.6}$$

$$n = \frac{n_0 v_0}{v_x} \; , \tag{5.7}$$

$$E_x = -\frac{m}{e}\,\omega_0 (b\cos\omega_0\eta - a\sin\omega_0\eta) \; , \tag{5.8}$$

where a and b are integration constants.

The transformation given by Eq. (5.4), becomes

$$x = v_0\eta + \frac{1}{\omega_0}\{a\sin\omega_0\eta + b(1-\cos\omega_0\eta)\} \; . \tag{5.9}$$

If we choose the origin such that $E_x=0$ at $x=\eta=0$, then b=0 and Eq. (5.9) is the same as the second line of Eq. (2.24) if we replace x_0 by $v_0\eta$, k by ω_0/v_0 and Ω by a/v_0. We can then use the inversion formulae, Eqs. (2.27) and (2.29), and obtain for E_x,

$$E_x(x) = \frac{m}{e}\,\omega_0 2 v_0 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} J_n\left(\frac{na}{v_0}\right) \sin\frac{n\omega_0 x}{v_0} \; . \tag{5.10}$$

Substituting this into Eq. (5.3) it follows that

$$n(x) = n_0 - 2n_0 \sum_{n=1}^{\infty} (-1)^n J_n\left(\frac{na}{v_0}\right) \cos\frac{n\omega_0 x}{v_0} \; . \tag{5.11}$$

Writing

$$v_x = \sum_{n=0}^{\infty} a_n \cos nx' \; ,$$

where

$$x' = \frac{\omega_0 x}{v_0} \quad , \qquad \eta' = \omega_0 \eta \quad ,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} v_x(x')\cos nx' dx'$$

$$= \frac{1}{\pi}\int_0^{2\pi} v_x(\eta')\cos nx'(\eta') \frac{dx'}{d\eta'} d\eta', \; n \neq 0 \; ,$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} v_x(\eta') \frac{dx'}{d\eta'} d\eta' \quad ,$$

we find after some algebra

$$v_x(x) = v_0 + \frac{a^2}{2v_0} - \sum_{n=1}^{\infty} \frac{2a}{n} (-1)^n J_n'\left(\frac{na}{v_0}\right) \cos \frac{n\omega_0 x}{v_0} \; , \tag{5.12}$$

where J_n' is the derivative of J_n with respect to its argument.

Some interesting results are implied by the above analysis. In Section II the calculations were only valid for $\Omega < 1$. In a similar way the validity of the present analysis requires $a < v_0$. It then follows from Eq. (5.6) and b=0 that v_x nowhere changes sign. Consequently, no electrons are trapped. The condition $a < v_0$, which is necessary to ensure the uniqueness of the Lagrangian transformation, excludes trapped electrons from the formalism. This is not surprising since the coexistence of trapped and free electrons cannot be described by the one-stream cold plasma model.

It should also be noted that the solutions (5.6), (5.7) and (5.8) cannot be obtained by "freezing" the solutions of the time-dependent problem in Section II, i.e., at no instant of time do the solutions in Section II, properly generalized to include the background flow velocity, take the spatial form of a cold B.G.K. wave. Furthermore, it is remarkable that in a cold plasma only periodic B.G.K. waves of a very special form, given in Eqs. (5.10), (5.11) and (5.12), are possible. This is in sharp contradistinction with the situation in a hot plasma,[8] and is due to the absence of trapped particles. In addition, it should be noted that the spatially averaged electron velocity does not equal v_0 but rather $v_0 + a^2/2v_0$ as is seen from Eq. (5.12).

(b) Electromagnetic Problem

Having obtained the solutions to the electrostatic problem we now consider the stationary form of the equations (4.5)-(4.7) treating $n(x)$ and $v_x(x)$ as known quantities through Eqs. (5.11) and (5.12). It follows that

$$\frac{dE_y}{dx} = 0 \quad , \tag{5.13}$$

$$v_x \frac{d}{dx}\left(v_y - \frac{e}{mc} A_y\right) = -\frac{e}{m} E_y \; , \tag{5.14}$$

$$\frac{d^2 A_y}{dx^2} - \frac{4\pi e n(x)}{c} v_y = 0 \; , \tag{5.15}$$

where A_y is the y-component of the vector potential. Choosing $E_y=0$ and $v_y = \frac{e}{mc} A_y$ in order to satisfy Eqs. (5.13) and (5.14), we obtain ($n=n_0N$)

$$\frac{d^2 A_y}{dx^2} - \frac{\omega_0^2}{c^2} N(x) A_y = 0 \; , \tag{5.16}$$

i.e., a Hill equation in space. In general the solutions of Eq. (5.16) are growing in space because of the minus sign. For special values of the Fourier coefficients of $N(x)$, however, the solutions can be "stable"; see Fig. 2 in Section IV. It may be expected, as in Section IV that growing solutions are limited by the magnetic force which was neglected in the longitudinal part of the problem.

(c) Special Case of the Complete Problem

If we consider the background to be at rest, i.e., $v_0=0$, but retain all magnetic force terms, it follows from Eqs. (4.1)-(4.7) that

$$v_x = 0 \; , \quad E_y = 0 \; , \tag{5.17}$$

$$\frac{dE_x}{dx} = -4\pi e(n-n_0) \ , \tag{5.18}$$

$$E_x + \frac{1}{c} v_y B_z = 0 \ , \tag{5.19}$$

and

$$4\pi e n v_y = c \frac{dB_z}{dx} \ . \tag{5.20}$$

Eqs. (5.18)-(5.20) permit a non-trivial class of solutions. If one of the functions $n(x)$, $E_x(x)$, $v_y(x)$ or $B_z(x)$ is prescribed the others can be calculated. As an example we consider

$$n = n_0(1+\Delta\cos kx) \ , \tag{5.21}$$

which gives

$$E_x = -\frac{4\pi e n_0 \Delta}{k} \sin kx \ , \tag{5.22}$$

and

$$B_z = \left[B_0^2 - \frac{8\pi^2 e^2 n_0^2 \Delta}{k^2} (4\cos kx + \Delta\cos 2kx)\right]^{1/2} \ , \tag{5.23}$$

where B_0^2 enters as an integration constant that must obviously satisfy the inequality

$$B_0^2 \geq \frac{8\pi^2 e^2 n_0^2 \Delta}{k^2} (4+\Delta) \ .$$

Finally, we find for v_y

$$v_y = c\,\frac{4\pi e n_0 \Delta}{k} \frac{\sin kx}{\left[B_0^2 - \frac{8\pi^2 e^2 n_0^2 \Delta}{k^2} (4\cos kx + \Delta\cos 2kx)\right]^{1/2}} \ . \tag{5.24}$$

In order to justify the non-relativistic treatment there is the additional requirement

$$\frac{4\pi e n_0 \Delta}{k B_0} \ll 1 \ .$$

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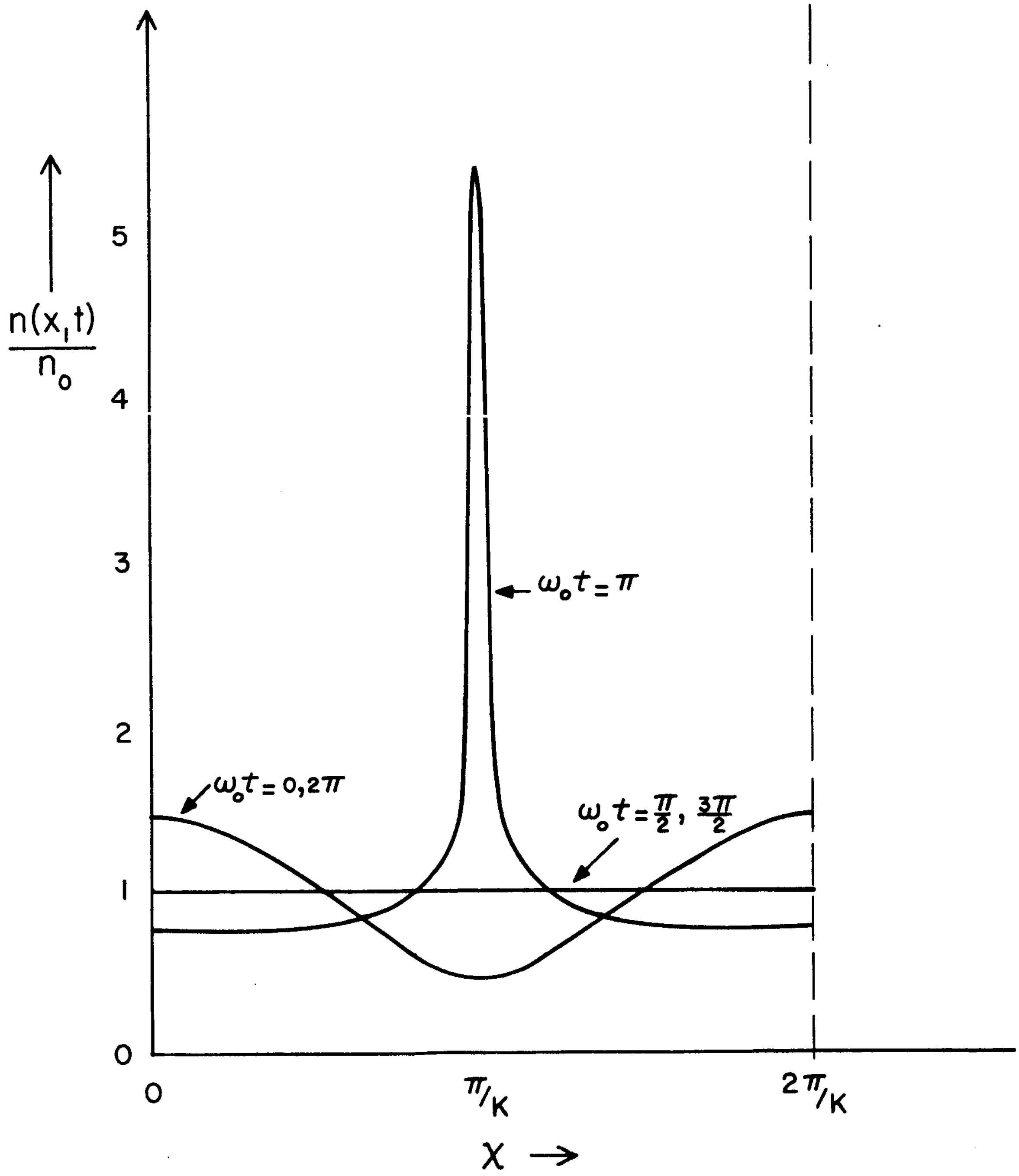
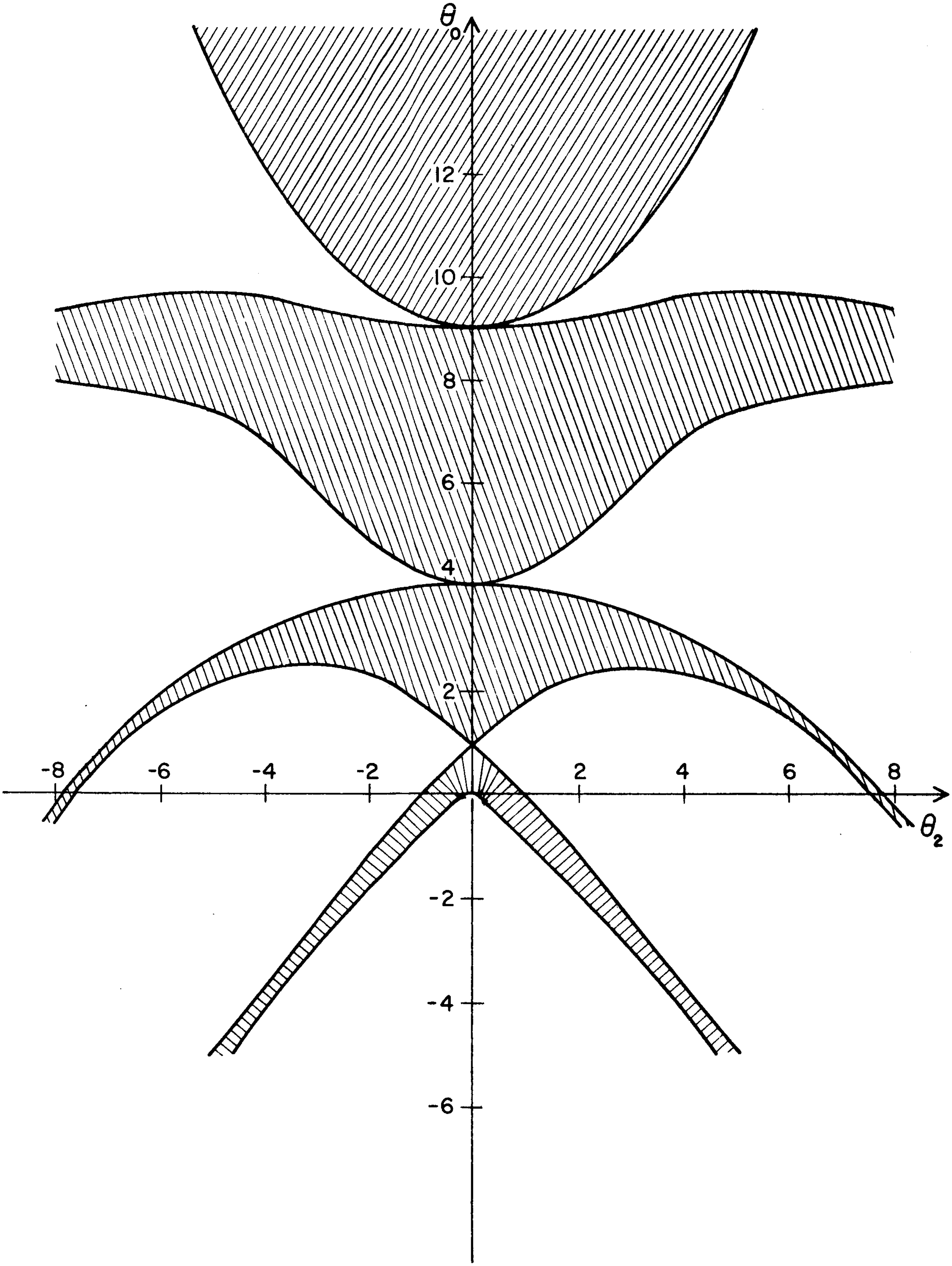
FIGURE CAPTIONS

Fig. 1. Density profiles with $\Delta = .45$.

Fig. 2. Stability diagram for Mathieu Equation according to Ref. 6 and 7. Stable regions are shaded.



n(x,t)/n₀
5
4
3
2
1
0
ωₒt = π
ωₒt = 0, 2π
ωₒt = π/2, 3π/2
0
π/K
2π/K
X →



θ_0
12
10
8
6
4
2
-8
-6
-4
-2
2
4
6
8
θ_2
-2
-4
-6